Notes: Unramified Sheaves of Sets

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1 Unramified sheaves of Sets

Let $\mathrm{Sm}_k^{\mathrm{Sm}}$ be the category of smooth k-schemes with smooth morphisms. We recall that $\mathcal{F}_k$ is the category of fields extensions of k with finite transcendance degree. The aim of this section is to define and study a class of presheaves that covers a large number of sheaves relevant to $\mathbb{A}^1$-homotopy.

Definition 1 *Let $\mathcal{S}$ be a presheaf of sets on Sm_k (or $\mathrm{Sm}_k^{\mathrm{Sm}}$). We say that $\mathcal{S}$ is unramified if the following three conditions hold :*

- **(U0)** *For any $X \in \mathrm{Sm}_k$ with irreducible components X_α's, $\alpha \in X^{(0)}$. The map $\mathcal{S}(X) \to \prod_{\alpha \in X^{(0)}} \mathcal{S}(X_\alpha)$ is a bijection.*
- **(U1)** *For any $X \in \mathrm{Sm}_k$ and any open subset $U \subset X$, if U is everywhere dense in X, then $\mathcal{S}(X) \to \mathcal{S}(U)$ is injective.*
- **(U2)** *For any $X \in \mathrm{Sm}_k$, irreducible with function field F, $\mathcal{S}(X) \to \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ is a bijection (where the intersection is taken in $\mathcal{S}(F)$).*

Let us make a few remarks about this definition.

1. Since all the morphisms that happens in the definition are smooth, a presheaf of set on Sm_k is unramified if and only if it is unramified as a presheaf of set on $\mathrm{Sm}_k^{\mathrm{Sm}}$. We will make use of this fact later.
2. Unramified presheaves are sheaves for the Zariski topology. Indeed, by **(U0)**, we can assume our scheme is irreducible, then, for any open U, we have the formula $\mathcal{S}(U) \cong \bigcap \lim_{x \in U^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$, and we can check that this formula defines a sheaf for the Zariski topology.
3. There is another description of such presheaves. The axiom **(U2)** can be replaced by the following, which we call **(U2')** : $\mathcal{S}$ is a sheaf for the Zariski topology and for any $X \in \mathrm{Sm}_k$, for any U open in X such that $X \setminus U$ is everywhere of codimension ≥ 2, then $\mathcal{S}(X) \to \mathcal{S}(U)$ is an isomorphism. Indeed if $\mathcal{S}$ is unramified, it is enough to check this in the case of an irreducible X. By **(U1)**, $\mathcal{S}(X) \to \mathcal{S}(U)$ is injective, by the assumption on U, U contains all the codimension 1 points of X, thus the composite $\mathcal{S}(X) \hookrightarrow \mathcal{S}(U) \cong \bigcap_{x \in U^{(1)}} S(\mathcal{O}_{X,x}) = \bigcap_{x \in X^{(1)}} S(\mathcal{O}_{X,x})$ is an isomorphism, so the first map must be an isomorphism as well. Conversely, if a sheaf verifies **(U0)** and **(U1)** and **(U2')**, it suffices to show that $\mathcal{S}(X) \hookrightarrow \bigcap_{x \in X^{(1)}} S(\mathcal{O}_{X,x})$ is surjective. Given a class $< U, f >$ in $\bigcap_{x \in X^{(1)}} S(\mathcal{O}_{X,x})$ (which is a subset of $\mathcal{S}(F)$), we can assume every $x \in X^{(1)}$ is in U and we can lift $< U, f >$ to an element in $\mathcal{S}(U)$ (because $\mathcal{S}$ is a sheaf), which is isomorphic to $\mathcal{S}(X)$ by assumption.

Remark If $\mathcal{S}$ is a sheaf of set in $\mathrm{Sm}_k^{\mathrm{Sm}}$ and $K \in \mathcal{F}_k$, then one can pullback $\mathcal{S}$ to $\mathrm{Sm}_K^{\mathrm{Sm}}$ and call the resulting sheaf the extension of $\mathcal{S}$ to K, moreover, if $\mathcal{S}$ is unramified, then its extension is unramified as well (we can prove it using **(U2')**). Beware that K might not be a perfect field anymore, so the equivalences we will prove below will not hold for the extended sheaves.

Let us give an important example of unramified sheaves :

Proposition 1.1 *If a sheaf $\mathcal{S}$ is strictly $\mathbb{A}^1$-local, then $\mathcal{S}$ is unramified.*

Proof Consider the coniveau spectral sequence associated to the sheaf :

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_x^{p+q}(X_{(x)}, \mathcal{S}) \Rightarrow H^{p+q}(X, \mathcal{S})\,.$$

We will admit that a strictly $\mathbb{A}^1$-invariant sheaf satisfies the following form of the purity theorem : for any $x \in X^{(p)}$,

$$H_x^{p+q}(X_{(x)}, \mathcal{S}) = \begin{cases} \mathcal{S}_{-p}(\kappa(x)) & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{cases}\,.$$

The key point here is that the coniveau spectral sequence is concentrated on the line $q = 0$. Hence it degenerates at the E_2 page and writing the start of the sequence gives an exact sequence

$$0 \to \mathcal{S}(X) \to \mathcal{S}(K(X)) \to \bigoplus_{x \in X^{(1)}} H_x^1(X_{(x)}, \mathcal{S}) \to 0\,.$$

That the sheaf satisfies **(U1)** can be seen by functoriality of the coniveau spectral sequence and of the exact sequence above, which yields a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{S}(X) & \longrightarrow & \mathcal{S}(K(X)) \\ \downarrow & & \downarrow & & \downarrow \mathrm{Id} \\ 0 & \longrightarrow & \mathcal{S}(U) & \longrightarrow & \mathcal{S}(K(X)) \end{array}$$

with exact rows.

For **(U2)**, we need to compute the kernel $\mathcal{S}(K(X)) \to \bigoplus_{x \in X^{(1)}} H_x^1(X_{(x)}, \mathcal{S})$. But this kernel is the intersection of the kernels $\mathcal{S}(K(X)) \to H_x^1(X_{(x)}, \mathcal{S})$. But since one has an exact sequence

$$0 = H_x^0(X_{(x)}, \mathcal{S}) \to \mathcal{S}(X_{(x)}) \to \mathcal{S}(X_{(x)} \setminus \{x\}) \to H_x^1(X_{(x)}, \mathcal{S})\,.$$

But since x has codimension 1, $\mathcal{S}(X_{(x)} \setminus \{x\}) = \mathcal{S}(K(X))$. So this kernel is $\mathcal{S}(X_{(x)}) = \mathcal{S}(\mathcal{O}_{X,x})$. So we find that $\mathcal{S}(X) = \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ as expected. □

The goal of this section is to describe in a more convenient way unramified sheaves. Recall that a functor $\mathcal{F}_k \to \mathbf{Set}$ is said to be continuous if $\mathcal{S}(F)$ is the filtering colimit of the values of $\mathcal{S}$ at the finite type subfields of F. We introduce the following objects :

Definition 2 *Given the following data :*

(D1) *A continuous functor $\mathcal{S} : \mathcal{F}_k \to \mathbf{Set}$.*

(D2) *For any $F \in \mathcal{F}_k$, and for any discrete valuation v on F, a subset $\mathcal{S}(\mathcal{O}_v) \subset \mathcal{S}(F)$.*

We say such data constitutes an unramified $\mathcal{F}_k^{\mathrm{Sm}}$-datum *if it is subject to the following axioms :*

(A1) *Let $E \subset F$ be an extension of field in $\mathcal{F}_k$, let v be a discrete valuation on E which restricts to a discrete valuation w on F, such that the ramification index $e(v/w)$ is 1. Then $\mathcal{S}(i)$ maps $\mathcal{S}(\mathcal{O}_w)$ in $\mathcal{S}(\mathcal{O}_v)$.I f moreover $\kappa(w) \to \kappa(v)$ is an isomorphism, then the following square of sets is cartesian :*

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \longrightarrow & \mathcal{S}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathcal{S}(E) & \longrightarrow & \mathcal{S}(F)\,. \end{array}$$

(A2) *Let X be smooth irreducible over k with function field F, then if $x \in \mathcal{S}(F)$, x lies in all but a finite number of subsets $\mathcal{S}(\mathcal{O}_y)$ wher $y \in X^{(1)}$.*

This definition is motivated by the following fact :

Proposition 1.2 *The category of unramified $\mathcal{F}_k^{\mathrm{Sm}}$-data is equivalent to the category of unramified sheaves of sets for the Nisnevich topology on $\mathrm{Sm}_k^{\mathrm{Sm}}$.*

Proof Let us first construct a functor from the category of unramified sheaves of sets on $\mathrm{Sm}_k^{\mathrm{Sm}}$ to the category of unramified $\mathcal{F}_k^{\mathrm{Sm}}$-data. This functor is simply the restriction of a presheaf $\mathcal{S}$ to the category of sheaves on $\mathcal{F}_k$. Let $\mathcal{S}$ be an unramified sheaf of sets on $\mathrm{Sm}_k^{\mathrm{Sm}}$. Continuity of the functor comes directly from the definition of the evaluation of $\mathcal{S}$ on an essentially smooth scheme. For a discrete valuation v on a field F, $\mathcal{S}(\mathcal{O}_v)$ is a subset of $\mathcal{S}(F)$, indeed, reduce first to the case where F is finitely generated over k. Pick a smooth irreducible model X for $\mathcal{O}_v$, Then $\mathcal{S}(\mathcal{O}_v)$ is the colimit over all opens of X containing v, this injects into $\mathcal{S}(F)$ which is the colimit of all open subsets of X. For a general F, we take the colimit over subextensions of F that are finitely generated.
Let us check **(A1)**, assume first that E and F are both of finite type over k, then, in the situation of **(A1)**, the following diagram of essentially smooth k-algebras is commutative

$$\begin{array}{ccc} \mathcal{O}_w & \longrightarrow & \mathcal{O}_v \\ \downarrow & & \downarrow \\ E & \underset{i}{\hookrightarrow} & F\,. \end{array}$$

Thus by applying $\mathcal{S}$, we find the first condition by functoriality of $\mathcal{S}$. In the general case, F is the colimit of its subfields of finite type over k, passing to the colimit in the previous diagram yields the first part of **(A1)**.
For the second part, assume that $\bar{i}$ is an isomorphism. Since the ramification $e(v/w)$ is 1 and the residue extension is trivial hence separable, the map $\mathrm{Spec}(\mathcal{O}_w) \to \mathrm{Spec}(\mathcal{O}_v)$ is étale ([Sta19, Lemma 09E7]). Since the residue extension is an isomorphism, the following square is a Nisnevich distinguished square over $\mathrm{Spec}(\mathcal{O}_w)$:

$$\begin{array}{ccc} \mathrm{Spec}(F) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(E) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_w)\,. \end{array}$$

All elements of this square are essentially smooth k-scheme. By a standard reindexing theorem ([AM69], A.3.3), we can assume they are all the projective limit over a common projective system I. Let us say the systems are respectively (F_λ), (E_λ), (V_λ), (W_λ) for $\mathrm{Spec}(E)$, $\mathrm{Spec}(F)$, $\mathrm{Spec}(\mathcal{O}_v)$, $\mathrm{Spec}(\mathcal{O}_w)$ respectively, and that the morphisms are all compatible with this system. Permanence theorems for inverse limits ([GD66] 8.10.5 for open and closed immersions, fiber products and isomorphisms and [GD67] 17.7.6 for étaleness) of morphisms show that for sufficiently large λ, the following square of smooth k-scheme is a Nisnevich distinguished square as well :

$$\begin{array}{ccc} F_\lambda & \longrightarrow & V_\lambda \\ \downarrow & & \downarrow \\ E_\lambda & \longrightarrow & W_\lambda\,. \end{array}$$

Then, since $\mathcal{S}$ is a Nisnevich sheaf by assumption, the following square is cartesian :

$$\begin{array}{ccc} \mathcal{S}(W_\lambda) & \longrightarrow & \mathcal{S}(V_\lambda) \\ \downarrow & & \downarrow \\ \mathcal{S}(E_\lambda) & \longrightarrow & \mathcal{S}(F_\lambda)\,. \end{array}$$

Then, we can take the filtered colimit over the indices $\mu \geq \lambda$, filtered colimits preserve finite limits hence the colimit square is cartesian as well, and said square is exactly

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \longrightarrow & \mathcal{S}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathcal{S}(E) & \longrightarrow & \mathcal{S}(F)\,. \end{array}$$

Let us check **(A2)** : let $x \in \mathcal{S}(F)$. By definition, x comes from some $x \in \mathcal{S}(U)$ for some open U of X. Then x is in all the $\mathcal{S}(\mathcal{O}_{X,y})$ for $y \in U^{(1)}$. Since X is irreducible, there are only a finite number of $y \in X^{(1)}$ that are not in $U^{(1)}$, thus **(A2)** is satisfied, and $\mathcal{S}_{|\mathcal{F}_k}$ constitutes an unramified $\mathcal{F}_k^{\mathrm{Sm}}$-datum.

Now let us go in the other direction, and construct an unramified sheaf of sets on $\mathrm{Sm}_k^{\mathrm{Sm}}$ from an unramified $\mathcal{F}_k^{\mathrm{Sm}}$-datum. Let $\mathcal{S}$ be such a data.
Let $X \in \mathrm{Sm}_k$ be irreducible with function field F. Define $\mathcal{S}(X)$ to be $\bigcap\limits_{x\in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$. For a non-irreducible X, extend this definition so that **(U0)** is satisfied. This defines $\mathcal{S}$ on objects. Let $f : Y \to X$ be a smooth morphism between two smooth k-schemes of finite type. We can assume X and Y are irreducible and that f is dominant. Let E and F be the function field of X and Y respectively. Then, define $\mathcal{S}(f) : \mathcal{S}(X) \to \mathcal{S}(Y)$ as the restriction of $\mathcal{S}(f) : \mathcal{S}(E) \to \mathcal{S}(F)$ to the subset $\bigcap_{x\in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$. We must show that this maps into $\bigcap_{y\in X^{(1)}} \mathcal{S}(\mathcal{O}_{Y,y})$. Since f is smooth and dominant, if $x \in X^{(1)}$, then $f^{-1}(x)$ is a finite subset of codimension-1 points, and the induced maps $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ correspond to an unramified extension of discrete valuation rings. If $x \in \bigcap\limits_{x\in X^{(1)}} \mathcal{O}_{X,x}$, then $x \in \bigcap\limits_{y\in Y^{(1)}} \mathcal{O}_{X,f(y)}$, which is mapped in $\bigcap\limits_{y\in Y^{(1)}} \mathcal{O}_{Y,y}$ by **(A1)**. Hence the map is well-defined.

It remains to check that this is a sheaf for the Nisnevich topology, and that it is unramified. Let us first show that this is a sheaf for the Nisnevich topology. We must show that $\mathcal{S}(\varnothing) = \{.\}$ and that, if $(U, p : V \to X)$ is an elementary Nisnevich covering, then, the following square is cartesian :

$$
\begin{CD}
\mathcal{S}(X) @>>> \mathcal{S}(V) \\
@VVV @VVV \\
\mathcal{S}(U) @>>> \mathcal{S}(U \times_X V)\,.
\end{CD}
$$

For the first part, we didn't say anything about the case of the empty set, so we might as well define $\mathcal{S}(\varnothing)$ to be a singleton. For the second part, let $(U, p : V \to X)$ be an elementary Nisnevich covering. Consider a diagram

$$
\begin{tikzcd}
X \arrow[rrd, bend left] \arrow[rd, bend right, "p"'] & & \\
& \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x}) \arrow[r] \arrow[d, hook] & \bigcap_{v \in V^{(1)}} \mathcal{S}(\mathcal{O}_{V,v}) \arrow[d] \\
& \bigcap_{x \in U^{(1)}} \mathcal{S}(\mathcal{O}_{U,x}) \arrow[r] & \bigcap_{v \in (U \times_X V)^{(1)}} \mathcal{S}(\mathcal{O}_{U \times_X V, v})\,.
\end{tikzcd}
$$

We must define an unique morphism $X \to \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ making the whole diagram commute. To do so, we must show the morphism $p : X \to \bigcap_{x \in U^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ factors to $\bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$. Let x be in $X^{(1)} \setminus U^{(1)}$, and let us show that $p(X) \subset \mathcal{S}(\mathcal{O}_{X,x})$. Since x is not in U, x is in $Z = X \setminus U$ which is isomorphic to $V \times_X Z$ with its reduced structure. Thus there is a point $y \in V$ which is above x with isomorphic residue fields. Let F' be the function field of V, and F the function field of X. Then $\mathcal{O}_{V,y}$ and $\mathcal{O}_{X,x}$ are discrete valuation ring, the valuation of $\mathcal{O}_{V,y}$ restricts to the valuation of $\mathcal{O}_{X,x}$ on F and is unramified because p is étale. Thus we can use **(A1)** to conclude that the following square is cartesian :

$$
\begin{CD}
\mathcal{S}(\mathcal{O}_{X,x}) @>>> \mathcal{S}(\mathcal{O}_{V,y}) \\
@VVV @VVV \\
\mathcal{S}(F) @>>> \mathcal{S}(F')\,.
\end{CD}
$$

Thus, by applying the cartesian property to $X \to \bigcap_{x \in U^{(1)}} \mathcal{S}(\mathcal{O}_{X,x}) \hookrightarrow \mathcal{S}(F)$ and to $X \to \bigcap_{v \in V^{(1)}} \mathcal{S}(\mathcal{O}_{V,v}) \hookrightarrow \mathcal{S}(F')$, we can factor p through $\mathcal{S}(\mathcal{O}_{X,x})$ as was to be shown. Doing so for every $x \in X^{(1)}$ that is not in $U^{(1)}$, p factors through $\bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$, and the fact that we are dealing with monomorphisms shows it factors uniquely. Hence, $\mathcal{S}$ defines a Nisnevich sheaf of sets.

By construction, it satisfies **(U0)**. Let U be everywhere dense in X, we can once again assume X is irreducible (so that U is just any nonempty open subset of X), then, the morphism $\mathcal{S}(X) \to \mathcal{S}(U)$ is $\bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x}) \to \bigcap_{x \in U^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ which certainly is injective. Finally, the very construction of $\mathcal{S}$ is so that it satisfies **(U2)**.

Let us check that this yields an inverse to the restriction functor. The fact that restriction followed by this composition is isomorphic to the identity is clear. Conversely, let $\mathcal{S}$ be an unramified $\mathcal{F}_k^{\mathrm{Sm}}$-datum, and let $\tilde{\mathcal{S}}$ be the datum obtained by the composition of the above construction followed by restriction. Then, for $F \in \mathcal{F}_k$, one has $\tilde{\mathcal{S}}(F) = \varinjlim \tilde{\mathcal{S}}(F_\alpha)$ where F_α runs over the subfields of F of same transcendence degree, smooth over k. For such field, we claim that $\tilde{\mathcal{S}}(F_\alpha) = \mathcal{S}(F_\alpha)$. Indeed, If F_α is of transcendence degree 0 over k, then F_α is also a smooth k-scheme and then $\mathcal{S}(F_\alpha) = \tilde{\mathcal{S}}(F_\alpha)$, if the transcendence degree is stricly positive, let X be smooth irreducible with function field F_α, then by **(A2)** any element in $\tilde{\mathcal{S}}(F_\alpha)$ is in all but a finite number of $\tilde{\mathcal{S}}(\mathcal{O}_{X,x})$, but those are equal to $\mathcal{S}(\mathcal{O}_{X,x})$ by construction. Hence $\tilde{\mathcal{S}}(F_\alpha) \subset \mathcal{S}(F_\alpha)$. By the same argument, one has $\tilde{\mathcal{S}}(F_\alpha) \supset \mathcal{S}(F_\alpha)$. □

We have given the data necessary to reconstruct an unramified sheaf of set on $\mathrm{Sm}_k^{\mathrm{Sm}}$, the next step is to add the data necessary to reconstruct an unramified sheaf of set on Sm_k.

Definition 3 *An* unramified $\mathcal{F}_k$-datum *is the data of an unramified $\mathcal{F}_k^{\mathrm{Sm}}$-datum, together with the additional data*

(D3) *For any $F \in \mathcal{F}_k$ and any discrete valuation v, a specialization map $s_v : \mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$.*

such that the following axioms are satisfied :

(A3) **(i)** *If $i : E \subset F$ is an extension in $\mathcal{F}_k$, v a discrete valuation on F that restricts to a discrete valuation w on E. Then, $\mathcal{S}(i)$ maps $\mathcal{S}(\mathcal{O}_w)$ in $\mathcal{S}(\mathcal{O}_v)$ and we have a commutative diagram :*

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \longrightarrow & \mathcal{S}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathcal{S}(\kappa(w)) & \longrightarrow & \mathcal{S}(\kappa(v)) . \end{array}$$

(ii) *If $i : E \subset F$ is an extension in $\mathcal{F}_k$ and v is a discrete valuation that restricts to 0 on E, then $\mathcal{S}(i)(\mathcal{S}(E)) \subset \mathcal{S}(\mathcal{O}_v)$ and if $j : E \subset \kappa(v)$ is the induced extension, then we have a commutative diagram :*

$$\begin{array}{ccc} \mathcal{S}(E) & & \\ {\scriptstyle \mathcal{S}(i)}\downarrow & \overset{\mathcal{S}(j)}{\searrow} & \\ \mathcal{S}(\mathcal{O}_v) & \xrightarrow[s_v]{} & \mathcal{S}(\kappa(v)) . \end{array}$$

(A4) **(i)** *For any X local of dimension 2 essentially smooth over k, with closed point $z \in X^{(2)}$ and for any $y_0 \in X^{(1)}$ such that $\bar{y_0}$ is also essentially smooth over k, then $s_{y_0} : \mathcal{S}(\mathcal{O}_{y_0}) \to \mathcal{S}(\kappa(y_0))$ maps $\bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y)$ into $\mathcal{S}(\mathcal{O}_{\bar{y_0},z})$.*

(ii) *The composition*

$$\bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y) \to \mathcal{S}(\mathcal{O}_{\bar{y_0},z}) \to \mathcal{S}(\kappa(z))$$

does not depend on y_0.

As previously, let us check that an unramified sheaf of set on Sm_k defines an unramified $\mathcal{F}_k$-datum. We already know it defines a $\mathcal{F}_k^{\mathrm{Sm}}$-datum. Let $F \in \mathcal{F}_k$ and v a valuation on F. We can define s_v by picking a smooth model. Once again we may assume F is of finite type. Let X be an irreducible smooth k-scheme with function field F a point x of codimension 1 such that $\mathcal{O}_{X,x} = \mathcal{O}_v$. Then the closed subset $Z = \overline{\{x\}}$ is smooth over k on an open dense subset U, so we might assume it is smooth as well, hence we can define $\mathcal{S}(O_v) \to \mathcal{S}(\kappa(v))$ by taking the colimit of the maps $\mathcal{S}(V) \to \mathcal{S}(V \cap Z)$ where V runs over the open subsets of X. Let us check that this data satifies **(A3)** and **(A4)**. Let us check **(A3)** first and let us keep the notation of **(i)**. We have the following diagram of essentially smooth k-schemes

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_v) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_w) \\ \big\uparrow & & \big\uparrow \\ \mathrm{Spec}(\kappa(v)) & \longrightarrow & \mathrm{Spec}(\kappa(w))\,. \end{array}$$

and we apply $\mathcal{S}$ to this diagram. Axiom **(ii)** is checked similarly.

Let us check **(A4)**. The image of $\mathcal{O}_{y_0}$ in X is everywhere dense, hence we have $\mathcal{S}(X) \subset \mathcal{S}(\mathcal{O}_{y_0})$ by **(U1)**. Moreover, if we consider the following diagram :

$$\begin{array}{ccc} \kappa(y_0) & \longrightarrow & \mathcal{O}_{y_0} \\ \big\downarrow & & \big\downarrow \\ \bar{y_0} & \longrightarrow & X\,, \end{array}$$

since $\kappa(y_0)$ is the function field of $\bar{y_0}$ which is of dimension one, $\mathcal{S}(\bar{y_0}) \subset \mathcal{S}(\kappa(y_0))$ by **(U1)** again. We have that the following diagram is commutative by applying $\mathcal{S}$ to the previous diagram :

$$\begin{array}{ccc} \mathcal{S}(X) & \hookrightarrow & \mathcal{S}(\mathcal{O}_{y_0}) \\ \big\downarrow & & \big\downarrow{\scriptstyle s_{y_0}} \\ \mathcal{S}(\bar{y_0}) & \hookrightarrow & \mathcal{S}(\kappa(y_0))\,. \end{array}$$

By **(U2)**, $\mathcal{S}(X) = \bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y)$. Commutativity of the above diagram shows that $\bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y)$ is mapped in $\mathcal{S}(\bar{y_0}) = \mathcal{S}(\mathcal{O}_{\bar{y_0},z})$. This shows the first part of **(A4)**.

For the second part of **(A4)**, the following diagram is commutative for all y_0 such that $\bar{y_0}$ is essentially smooth over k :

$$\begin{array}{ccc} & & \bar{y_0} \\ & \nearrow & \big\downarrow \\ \mathrm{Spec}(\kappa(z)) & \longrightarrow & X\,. \end{array}$$

By definition, applying $\mathcal{S}$ gives

$$\begin{array}{ccc} \mathcal{S}(X) & \longrightarrow & \mathcal{S}(\bar{y_0}) \\ & \searrow & \big\downarrow{\scriptstyle s_{y_0}} \\ & & \mathcal{S}(\kappa(z)) \end{array}$$

and $\mathcal{S}(X) = \bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y)$. Because it comes from the inclusion $\mathrm{Spec}(\kappa(z)) \to X$, the diagonal map is independent from y_0. This shows the second part of **(A4)**.

We can now state and prove the main theorem :

Theorem 1.3 *Given a* F_k*-datum* $\mathcal{S}$*, there is a unique way to extend the unramified sheaf on* $\mathrm{Sm}_k^{\mathrm{Sm}}$ *induced by the* $\mathcal{F}_k^{\mathrm{Sm}}$*-datum composing* $\mathcal{S}$ *such that for any* $F \in \mathcal{F}_k$ *discrete valuation ring* $\mathcal{O}_v$ *in* F *with separable residue field, the map* $\mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$ *induced by the extended sheaf coincide with the specialization map* s_v *of* $\mathcal{S}$*, and this extended sheaf is unramified.*

Proof The functor is already defined on object, it remains to define it on map. We will first define it in the case of closed immersions and then reduce to this case. To define it on closed immersion, we procede by induction on the dimension on the closed immersion. Let us first define it on closed immersion of codimension one. Let $i : Y \to X$ such an immersion. We may assume Y and X are both irreducible. In the general case, we define $s(i)$ as the product of the $\mathcal{S}(i_\alpha)$ where i_α are the map induced on the irreducible component, this reduction makes sense by **(U0)**. Let y be the generic point of Y, it is of codimension 1 in X by assumption. By **(U2)**, to define $s(i) : \mathcal{S}(X) \to \mathcal{S}(Y)$ in a functorial way, we must have

$$\begin{array}{ccc} \mathcal{S}(X) & \xrightarrow{s(i)} & \mathcal{S}(Y) \\ \downarrow & & \downarrow \\ \mathcal{S}(\mathcal{O}_{X,y}) & \xrightarrow[s_y]{} & \mathcal{S}(\kappa(y)) . \end{array}$$

We thus have to check that for all $z \in Y^{(1)}$, s_y maps $\mathcal{S}(X)$ in $\mathcal{S}(\mathcal{O}_{Y,z})$, since $\mathcal{S}(Y) = \bigcap_{z \in Y^{(1)}} \mathcal{S}(\mathcal{O}_{Y,z})$, this will define $s(i)$ in a unique way.

Let $z \in Y^{(1)}$, then z has codimension 2 in X and we can apply **(A4)** to $\mathcal{O}_{X,z}$ which is of dimension 2. It shows that $\mathcal{S}(\mathcal{O}_{X,z})$ (which contains $\mathcal{S}(X)$) is sent in $\mathcal{S}(\mathcal{O}_{\bar{y},z}) = \mathcal{S}(\mathcal{O}_{Y,z})$ as expected.

The next step is to define $s(i)$ where $i : Z \to X$ is a closed immersion of codimension d. The key point is the following lemma :

Lemma 1.4 *Let* $i : Z \to X$ *a closed immersion of codimension* $d > 0$*. Assume that i factors as*

$$Z \xrightarrow{j_1} Y_1 \xrightarrow{j_2} Y_2 \to \cdots \to Y_d = X ,$$

such that each consecutive Y_i *is closed and smooth over* X *and the* j_i *are closed immersions of codimension* 1*. Then, the composition*

$$\mathcal{S}(X) \xrightarrow{s(j_d)} \cdots \to \mathcal{S}(Y_2) \xrightarrow{s(j_2)} \mathcal{S}(Y_1) \xrightarrow{s(j_1)} Z$$

is independent of the factorization. This composition will be denoted $s(i)$

Proof There is nothing to prove if $d = 1$, so let us assume $d \geq 2$. We can further assume that Z is irreducible of generic point z. Since $\mathcal{S}$ is a sheaf, we can replace X by any neighborhood of z, and even replace X by the local ring $A = \mathcal{O}_{X,z}$. Let us do such a reduction. In this case, the case

$d = 2$ is exactly the one handled by **(A4)**. In the general case, since A is regular of dimension d, the sequence of closed immersion is of the form

$$\mathrm{Spec}(A/(x_1, \dots, x_d)) \to \mathrm{Spec}(A/(x_2, \cdots, x_d)) \to \cdots \to \mathrm{Spec}(A/(x_d)) \to \mathrm{Spec}(A)$$

where $(x_1, \dots, x_d)$ generate the maximal ideal $\mathfrak{m}$ of A. Smoothness of each successive member of the flag is equivalent ([GD67], 17.12.2) to the fact that $(x_1, \dots, x_d)$ is a regular sequence in A. We thus have to show that the composite

$$\mathcal{S}(\mathrm{Spec}(A/(x_1, \dots, x_d))) \to \mathcal{S}(\mathrm{Spec}(A/(x_2, \dots, x_d))) \to \cdots \to \mathcal{S}(\mathrm{Spec}(A/(x_d))) \to \mathcal{S}(\mathrm{Spec}(A))$$

does not depend on the regular sequence $(x_1, \dots, x_d)$. If $(x'_1, \dots, x'_d)$ is another regular sequence, then, since both reduce to a $\kappa(z)$-basis of $\mathfrak{m}/\mathfrak{m}^2$, there exists $M \in \mathrm{Gl}_d(A)$ sending $(x'_1, \dots, x'_d)$ to $(x_1, \dots, x_d)$. Multiplying x_1 by any unit of A does not change the flag of subvariety associated to the sequence, hence we may even assume that $M \in \mathrm{Sl}_d(A)$. It is known ([Knu91], VI, corollary 1.5.3) that the special linear group of a local ring is generated by elementary matrices.
By the case $d = 2$, we can freely permute x_i and x_{i+1} in the regular sequences we are considering, hence, we can freely permute all the elements of the regular sequences we are working with without changing the composition. Hence, after permuting the elements of the sequences accordingly, it remains to show that the regular sequences $(x_1, \dots, x_d)$ and $(x_1 + ax_2, x_2, \dots, x_d)$ induce the same composition : indeed by the induction hypothesis, once this is shown, all the remaining elementary matrices are actually acting on $(x_2, \dots, x_d)$ and the induction hypothesis will show that it is independent of the choice of the regular sequence for those variables.
But actually, $(x_1, \dots, x_d)$ and $(x_1 + ax_2, \dots, x_d)$ induce the same flag of subvarieties. So the composition are obviously the same, and thus, the lemma is proved. □

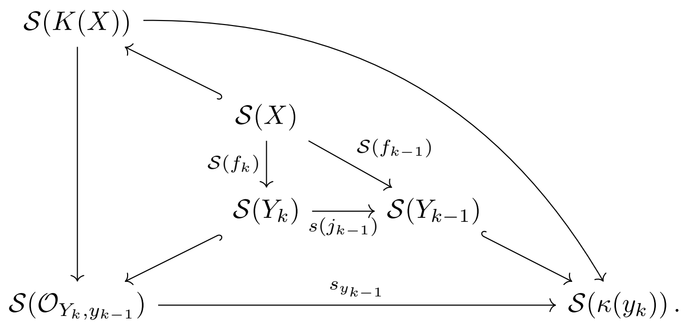
Back to the proof of the theorem. We have defined $s(i)$ when i is a closed immersion that splits as a composition of codimension 1. For a general closed immersion $Z \to X$, one can cover X by opens subsets U such that the immersion $Z \cap U \to U$ splits as above. The map $s(i)$ is then uniquely defined on each such opens that cover X and the maps are compatible so we can define $s(i) : \mathcal{S}(X) \to \mathcal{S}(Z)$ using that $\mathcal{S}$ is a Zariski sheaf. Now there only remains to define $\mathcal{S}(f)$ for an arbitrary morphism $f : Y \to X$ between smooth k-schemes. Without loss of generality, we can assume both schemes are separated (or even affine) since any smooth scheme has a Zariski covering by such objects. Indeed, once f is defined on such elements, they will be one and only one way to define it for arbitrary morphism by glueing because of the sheaf property of $\mathcal{S}$ with respect to open immersions. Then f can be written as the composition $Y \overset{\Gamma_f}{\hookrightarrow} Y \times_k X \overset{\pi}{\to} X$, the first map being a closed immersion, and the second being the smooth projection on X. Thus $\mathcal{S}(f)$ must be $s(\Gamma_f) \circ \mathcal{S}(\pi)$. For the time being, we denote it $s(f)$, for we have yet to show that it coincides with $\mathcal{S}(f)$ for smooth maps.
We will use the following technical lemma :

Lemma 1.5 *Let $f : Y \to X$ be a smooth map. Then, for all $x \in Y$, there an open neighborhood V of $f(x)$ in Y such that and an open U in X such that $f(U) \subset V$ and such that Γ_f factors on U as a composition $Y = Y_0 \overset{j_0}{\hookrightarrow} Y_1 \overset{j_1}{\hookrightarrow} \cdots \overset{j_{d-1}}{\hookrightarrow} Y_d = U \times_k V$ of codimension 1 closed immersions, with the following properties*

1. *The composition $f_k := Y_k \hookrightarrow Y_{k+1} \hookrightarrow \cdots Y_d = U \times_k V \to V$ is smooth.*
2. *The restriction on $K(X)$ of the valuation that Y_{j-1} induces on $K(Y_j)$ is zero.*

Proof The existence of the factorization by smooth varieties follows essentially by [GD67], 17.12.2 again. Only the part about the valuations should require some explanation. Our properties are local so we may assume that V is affine, say $V = \mathrm{Spec}(A)$. We can furthermore assume $U \times_k V$ is also affine, equal to $\mathrm{Spec}(B)$ where $B = A[X_1, \cdots, X_r]/(f_1, \cdots, f_s)$. Then, the immersion $Y \hookrightarrow U \times_k V$ is defined by a regular sequence $(f_{s+1}, \cdots, f_{s+d})$. The condition on valuation will be satisfied if we know each of the f_{s+k} are not in A, for in this case, the valuation on $\mathrm{Frac}(A)$ induced by $Y_{k-1} = \mathrm{Spec}(A[X_1, \ldots, X_r]/(f_1, \ldots, f_{s+k-1}))$ on $Y_k = \mathrm{Spec}(A[X_1, \ldots, X_r]/(f_1, \ldots, f_{s+k}))$ is simply viewing an element of $\mathrm{Frac}(A)$ as an element of $\mathrm{Frac}\left(A[X_1, \ldots, X_r]/(f_1, \ldots, f_{s+k}\right)$ and looking at its multiplicity on $\mathrm{Spec}(A[X_1, \ldots, X_r]/(f_1, \ldots, f_{s+k-1}))$. If all the f_{s+i} are not in A, then this valuation will always be zero. But by [GD67], 17.12.2, the family $(df_{s+1}, \ldots, df_{s+d})$ in $\Omega_{B/A}$ must be a free familly. Since $dg = 0$ if $g \in A$, none of the f_{s+k} is in A and the lemma is proved. □

Now, let us consider such a flag, and denote by y_k the generic point of Y_k. For every $1 \le k \le d$ we have the following diagram :



The existence of the leftmost map and the commutativity of the outermost triange comes from **(ii)** of **(A3)**. Commutativity of the outer squares are the definition of the maps involved. This shows that $s(j_{k-1}) \circ \mathcal{S}(f_k) = \mathcal{S}(f_{k-1})$. From that we deduce that

$$\begin{aligned} \mathcal{S}(f) &= \mathcal{S}(f_0) \\ &= s(j_0) \circ s(j_1) \circ \cdots \circ s(j_d) \circ \mathcal{S}(f_d)\,, \end{aligned}$$

but $s(j_0) \circ s(j_1) \circ \cdots \circ s(j_d) = s(\Gamma_f)$ by definition and $\mathcal{S}(f_d) = \mathcal{S}(\pi_X)$. So $\mathcal{S}(f) = s(\Gamma_f) \circ \mathcal{S}(\pi_X) = s(f)$ at least locally for the Zariski topology, and since $\mathcal{S}$ is a sheaf for the Zariski topology, they are the same, and we will not distinguish anymore.

Let us now check that the definiton we have for closed immersion and the already-defined value of $\mathcal{S}$ on smooth maps is compatible. Let $f : Y \to X$ be a smooth morphism and $i : Z \to X$ be a closed immersion of codimension 1, consider the cartesian square

$$\begin{array}{ccc} Z' & \xhookrightarrow{i'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xhookrightarrow[i]{} & X\,. \end{array}$$

As usual, we may assume that X, Y and Z are irreducible, then Z' is 1-codimensional. Then, compatibility comes from the commutativity of the following diagram :

$$\begin{array}{ccccccc}
\mathcal{S}(\mathcal{O}_z) & & & \xrightarrow{\quad s_z \quad} & & & \mathcal{S}(\kappa(z)) \\
 & \nwarrow & & & & \nearrow & \\
\big\downarrow & & \mathcal{S}(X) & \xrightarrow{s(i)} & \mathcal{S}(Z) & & \big\downarrow \\
 & & {\scriptstyle \mathcal{S}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{S}(f')} & & \\
 & & \mathcal{S}(Y) & \xrightarrow[s(i')]{} & \mathcal{S}(Z') & & \\
 & \swarrow & & & & \searrow & \\
\mathcal{S}(\mathcal{O}_{z'}) & & & \xrightarrow{\quad s_{z'} \quad} & & & \mathcal{S}(\kappa(z'))\,.
\end{array}$$

The existence of the left and right outermost maps, and the fact that the left and right trapezoids form a commutative diagram comes from **(i)** of axiom **(A3)** along with the definition of $\mathcal{S}$ on smooth maps. The fact that the outermost square commutes is also **(i)** of **(A3)**. The top and bottom trapezoids are merely the definition of $s(i)$ and $s(i')$.
It then follows that $s(i)$ is compatible with $\mathcal{S}(f)$ for any smooth morphism f and closed immersion i, indeed such an immersion can locally be factored as a sequence of codimension 1 closed immersions, and then a repeated application of the preceding case yields the compatibility.

The last point to check is functoriality of $\mathcal{S}$, Let $g : Z \to Y$ and $f : Y \to X$. It suffices to apply $\mathcal{S}$ to the following diagram :

$$\begin{array}{ccccc}
Z & \hookrightarrow & Z \times_k Y & \hookrightarrow & Z \times_k Y \times_k X \\
 & \searrow^{g} & \big\downarrow & & \big\downarrow \\
 & & Y & \hookrightarrow & Y \times_k X \\
 & & & \searrow^{f} & \big\downarrow \\
 & & & & X\,.
\end{array}$$

By what have been checked this far, the resulting diagram will be commutative. Commutativity of the resulting diagram is exactly the fact that $\mathcal{S}(f \circ g) = \mathcal{S}(g) \circ \mathcal{S}(f)$.

Finally, the object we defined is indeed an unramified sheaf, for these properties only depends on the restriction to $\mathrm{Sm}_k^{\mathrm{Sm}}$ as we have already remarked. $\square$

Remark This rather long proof also shows that, if $\mathcal{E}$ is a sheaf on Sm_k of set satisfying **(U0)** and **(U1)** and if $\mathcal{S}$ is an unramified sheaf of set. Then to construct a morphism $\Phi : \mathcal{E} \to \mathcal{S}$, it is enough to give a natural transform $\phi : \mathcal{E}_{|\mathcal{F}_k} \to \mathcal{S}_{|\mathcal{F}_k}$ such that for any $F \in \mathcal{F}_k$ and any discrete valuation v on F, such that ϕ sends $\mathcal{E}(\mathcal{O}_v) \subset \mathcal{E}(F)$ in $\mathcal{S}(\mathcal{O}_v) \subset \mathcal{S}(F)$, and such that the following square commute :

$$\begin{array}{ccc}
\mathcal{E}(\mathcal{O}_v) & \xrightarrow{s_v} & \mathcal{E}(\kappa(v)) \\
{\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\
\mathcal{S}(\mathcal{O}_v) & \longrightarrow & \mathcal{S}(\kappa(v))\,.
\end{array}$$

We now state a condition on which the resulting sheaves are $\mathbb{A}^1$-invariant.

Proposition 1.6 1) *Let $\mathcal{S}$ be an unramified sheaf of sets on $\mathrm{Sm}_k^{\mathrm{Sm}}$, then, $\mathcal{S}$ is $\mathbb{A}^1$-invariant if and only if the following is satisfied : for any k-smooth local ring A of dimension at most one, the canonical map $\mathcal{S}(A) \to \mathcal{S}(\mathbb{A}^1_A)$ is bijective.*

2) *Let $\mathcal{S}$ be an unramified sheaf of sets on Sm_k, then, $\mathcal{S}$ is $\mathbb{A}^1$-invariant if and only if it satisfies the following : for any $F \in \mathcal{F}_k$, the canonical map $\mathcal{S}(F) \to \mathcal{S}(\mathbb{A}^1_F)$ is bijective.*

Proof 1) One implication is clear. Let us assume $\mathcal{S}$ satisfies the condition. Let $X \in \mathrm{Sm}_k$ be irreducible with function field F. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}(X) & \longrightarrow & \mathcal{S}(\mathbb{A}^1_X) \\ \downarrow & & \downarrow \\ \mathcal{S}(F) & \longrightarrow & \mathcal{S}(F(T)) \end{array}$$

in which every map is injective ($\mathcal{S}(X) \to \mathcal{S}(\mathbb{A}^1_X)$ is injective since it admits a left inverse by considering $X \hookrightarrow \mathbb{A}^1_X \to X$).

The map $\mathcal{S}(\mathbb{A}^1_X) \to \mathcal{S}(F(T))$ factors as $\mathcal{S}(\mathbb{A}^1_X) \to \mathcal{S}(\mathbb{A}^1_F) \to \mathcal{S}(F(T))$. By assumption $\mathcal{S}(F) = \mathcal{S}(\mathbb{A}^1_F)$. so that $\mathcal{S}(\mathbb{A}^1_X)$ is actually a subset of $\mathcal{S}(F)$. Now for any $x \in X^{(1)}$, it is enough to prouve that there are inclusions $\mathcal{S}(\mathbb{A}^1_X) \subset \mathcal{S}(\mathcal{O}_{X,x}) \subset \mathcal{S}(F)$, indeed, taking intersection we would then have $\mathcal{S}(\mathbb{A}^1_X) \subset \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$. Since by assumption we have $\mathcal{S}(\mathcal{O}_{X,x}) = \mathcal{S}(\mathbb{A}^1_{\mathcal{O}_{X,x}})$, we're done.

2) As previously one implication is immediate. In the other direction, let $X \in \mathrm{Sm}_k$ be irreducible with function field F. Then we have a commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{A}^1_X) & \hookrightarrow & \mathcal{S}(\mathbb{A}^1_F) \\ \downarrow & & \downarrow \\ \mathcal{S}(X) & \hookrightarrow & \mathcal{S}(F) \end{array}$$

where the rightmost map is bijective. This implies that the leftmost map is injective. We also know that this map is surjective as it is a left inverse to $\mathcal{S}(X) \to \mathcal{S}(\mathbb{A}^1_X)$. Hence the map is bijective, which proves the claim.

□

Remark If we are given an unramified sheaf $\mathcal{S}$ on $\mathrm{Sm}_k^{\mathrm{Sm}}$, with added data **(D3)** to the associated $\mathcal{F}_k^{\mathrm{Sm}}$-datum, with the property that $\mathcal{S}(F) \to \mathcal{S}(F(T))$ is injective, then $\mathcal{S}$ is an unramified $\mathcal{F}_k$-datum if and only if its extension to $k(T)$ is an unramified $\mathcal{F}_{k(T)}$-datum. Indeed thanks to this hypothesis, all the maps $\mathcal{S}(X) \to \mathcal{S}(X \otimes_k k(T))$, $\mathcal{S}(\mathcal{O}_{X,x}) \to \mathcal{S}(\mathcal{O}_{X \otimes_k k(T), \bar{x} \otimes_k k(T)})$ etc.. will be injective. Hence, to check the axioms of an unramified $\mathcal{F}_{k(T)}$ datum for the extension of $\mathcal{S}$ will check it as well for $\mathcal{S}$. By this fact, one can often reduce to the case of an infinite base field.

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